(Def^n) Let H be a Hilbert space. An orthonormal set M of H is said to be complete if it is not contained in any other orthonormal set.

"or"

A maximal orthonormal set M of H is called a complete orthonormal set.

(Theorem) Let H be a Hilbert space and let $\{e_i\}$ be an orthonormal set in H. Then the following statements are equivalent.

(1) $\{e_i\}$ is complete

(2) $x \perp \{e_i\} \Rightarrow x = \theta$

(3) for every $x \in H$, $x = \sum_{i=1}^{\infty} (x, e_i) e_i$

(4) for every $x \in H$, $\|x\|^2 = \sum |(x, e_i)|^2$

Proof:- (1) $\Rightarrow$ (2)

Suppose that $\{e_i\}$ is a complete orthonormal set in H. Suppose ~~that~~ there exists a non zero $x \in H$ such that $x \perp \{e_i\}$. We put $e = \frac{x}{\|x\|}$. Then $\|e\| = 1$ and $\{e, e_i\}$ is an orthonormal set which properly contains $\{e_i\}$. This contradicts our assumption. This proves that $x = \theta$.

(2) $\Rightarrow$ (3)

Suppose that $x \perp \{e_i\} \Rightarrow x = \theta$. It is known that $\{x - \sum (x, e_i) e_i\} \perp e_j$ for every j.

So according the our assumption,

$$x - \sum (x, e_i) e_i = \theta$$

$$\text{i.e., } x = \sum (x, e_i) e_i$$

$(3) \Rightarrow (4)$

Suppose that for every $x \in H$, $x = \sum (x, e_i) e_i$

Then,

$$\|x\|^2 = (x, x)$$

$$= \left(\sum (x, e_i) e_i, \sum (x, e_j) e_j\right)$$

$$= \left(\lim_{n\to\infty} \sum_{i=1}^{n} (x, e_i) e_i, \lim_{n\to\infty} \sum_{j=1}^{n} (x, e_j) e_j\right)$$

$$= \lim_{n\to\infty} \left(\sum_{i=1}^{n} (x, e_i) e_i, \sum_{j=1}^{n} (x, e_j) e_j\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (x, e_i) \overline{(x, e_i)}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} |(x, e_i)|^2$$

$$= \sum_{i=1}^{\infty} |(x, e_i)|^2$$

$(4) \Rightarrow (1)$

Suppose that for every $x \in H$, $\|x\|^2 = \sum_{i=1}^{\infty} |(x, e_i)|^2$

If possible, suppose that $\{e_i\}$ is not complete.

This means that there is a non-zero element e, such that $\|e\| = 1$ and $\{e, e_i\}$ is orthonormal.

Now, $(e, e_i) = 0$ for all i.

$$\therefore \|e\|^2 = \sum |(e, e_i)|^2 = 0.$$

This gives $e = \theta$. Which contradicts the fact that $\|e\| = 1$. Hence $\{e_i\}$ is complete.

Theorem) A Hilbert space H is separable iff it contains a complete orthonormal set.

Proof:- Suppose that H is a separable space. Then H contains a countable set N which is dense in H. Now using Gram-Schmidt orthogonalisation process. We obtain a orthonormal sequence M in H. Let $x \perp M$ then $x \perp N$. Also N is dense in H. So for given $\varepsilon > 0$ there exists $y \in N$ such that $\|y - x\| < \varepsilon$. This gives,

$$\|x\|^2 = (x, x) = (x, x) - (y, x) = (x - y, x)$$
$$\leq \|x - y\| \|x\|$$
$$< \varepsilon \|x\|.$$

i.e. $\|x\| < \varepsilon$.

Since this is true for every $\varepsilon > 0$, $\|x\| = 0$. Hence $x = \theta$. This proves that M is a complete orthonormal set.

Conversely, suppose that $\{e_i\}$ is a complete orthonormal set. We consider the set N of all rational linear combinations of elements of $\{e_i\}$. Then N is a

Countable subset of H. Let $x \in H$ and $\varepsilon > 0$ be given. Since $x = \sum (x, e_i) e_i$, we can choose a positive integer n such that

$$\left\| x - \sum_{i=1}^{n} (x, e_i) e_i \right\| < \frac{\varepsilon}{2}$$

We approximate the complex number (x, e_i) by numbers of the form $\gamma_i^{(n)}$ such that,

$$\left\| \sum \{ (x, e_i) - \gamma_i^{(n)} \} e_i \right\| < \varepsilon/2 .$$

where $\gamma_i^{(n)} = \alpha_i^{(n)} + i \beta_i^{(n)}$, $\alpha_i^{(n)}$, $\beta_i^{(n)}$ having rational numbers. We put

$$y = \sum_{i=1}^{n} \gamma_i^{(n)} e_i$$ then,

$$\| x - y \| \leq \left\| x - \sum (x, e_i) e_i \right\| + \left\| \{ \sum (x, e_i) e_i - \gamma_i^{(n)} \} e_i \right\|$$

$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

Hence, x is a limit point of N since $y \in N$. Hence H is separable.

Proposition:- Any orthonormal set in an inner Product Space E (or in a Hilbert Space) is Contained in a maximal orthonormal set.

(A maximal orthonormal set is also known as a Complete orthonormal set).

Proof:- Let P be the set of all orthonormal sets in E Containing the given orthonormal set. We Partially order P by set inclusion relation $\subseteq$ so that $(P, \subseteq)$ is a Partially ordered set. Let $C = (O_\alpha)$ be any chain in P and let $O = \cup O_\alpha$. Then every $x \in O$ belongs to some O_α, hence $\|x\| = 1$. Furthermore, if $x, y \in O$, $x \neq y$, then $x \in O_\alpha$, $y \in O_\beta$ (Say) and if $O_\alpha \subset O_\beta$ then $x, y \in O_\beta$, thus $(x, y) = 0$ Consequently O is an orthonormal set Containing the given orthonormal set i.e. $O \in P$ and clearly O is an upper bound for the Chain in P. Therefore, by Zorn's lemma, P Possesses a maximal element. Hence there exists a maximal orthonormal set Containing the given orthonormal set.

Proposition:- Every non-zero Hilbert Space Contains a Complete orthonormal set.

Proof:- Let H be a non-zero Hilbert Space. Then there exists a non-zero element $x \in H$ and taking $e = \frac{x}{\|x\|}$, we find that the Singleton $\{e\}$ is an orthonormal set in H. Hence the class of all orthonormal sets in H is non-empty. Let p be the class of all orthonormal sets in H is non-empty. Let p be the class of all orthonormal sets in H. Then $(p, \subseteq)$ is a partially ordered set with respect to set inclusion relation, $\subseteq$. Let $C = (O_\alpha)$ be any chain in p and let $O = \cup O_\alpha$. Then every $x \in O$ belongs to some O_α, hence $\|x\| = 1$.

~~For~~ Furthermore, if $x, y \in O$. $x \neq y$, then $x \in O_\alpha$, $y \in O_\beta$ (say) and if $O_\alpha \subseteq O_\beta$ then $x, y \in O_\beta$ and hence $(x, y) = 0$. Thus O is an orthonormal set in H i.e. $O \in p$. and clearly O is an upper bound for the chain ~~o~~ in p. Hence, by Zorn's lemma, p possesses a maximal element i.e. there exists a maximal orthonormal set in H i.e. there exists a complete orthonormal set in H.

Properties of orthonormal sets.

Q No> If $\{e_1, e_2, \text{------}, e_n\}$ be a finite orthonormal set in a Hilbert space H and x be any vector in H, then

(a) $\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2$. [Bessel's inequality for finite orthonormal sets].

(b) $x - \sum_{i=1}^{n} (x, e_i) e_i \perp e_j$ for each j.

Proof:- (a) Let $\alpha_i = (x, e_i)$ for $i = 1, 2, ----, n$.

Then, $0 \leq \left\| x - \sum_{i=1}^{n} \alpha_i e_i \right\|^2$

$= \left(x - \sum_{i=1}^{n} \alpha_i e_i , x - \sum_{j=1}^{n} \alpha_j e_j \right)$

$= (x, x) - \sum_{j=1}^{n} \overline{\alpha_j} (x, e_j) - \sum_{i=1}^{n} \alpha_i (e_i, x) + \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} (e_i, e_j)$

$= \|x\|^2 - \sum_{j=1}^{n} \overline{\alpha_j} \alpha_j - \sum_{i=1}^{n} \alpha_i \overline{\alpha_i} + \sum_{i=1}^{n} \alpha_i \left(\sum_{j=1}^{n} \overline{\alpha_j} (e_i, e_j) \right)$

$= \|x\|^2 - \sum_{j=1}^{n} |\alpha_j|^2 - \sum_{i=1}^{n} |\alpha_i|^2 + \sum_{i=1}^{n} \alpha_i \overline{\alpha_i}$

[$\because (e_i, e_j) = 0$ for each j different from i]

$= \|x\|^2 - \sum_{i=1}^{n} |\alpha_i|^2$ [$\because \alpha_i \overline{\alpha_i} = |\alpha_i|^2$]

$\therefore \sum_{i=1}^{n} |\alpha_i|^2 \leq \|x\|^2$, and hence

$\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2$

(b) $\left(x - \sum_{i=1}^{n} (x, e_i) e_i , e_j \right)$

$= (x, e_j) - \sum_{i=1}^{n} (x, e_i)(e_i, e_j)$

$= (x, e_j) - (x, e_j)$

$= 0$. [$\because (e_i, e_j) = 0$ for each i different from j]

Hence, $x - \sum_{i=1}^{n} (x, e_i) e_i \perp e_j$ for each j.

This Completes the Proof.